J. Math. Anal. Appl. 375 (2011) 738–744

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On Marcinkiewicz–Zygmund laws

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ARTICLE INFO

Article history:
Received 5 May 2009
Available online 8 October 2010
Submitted by M. Peligrad

Keywords:
Strong laws of large numbers
φ-Mixing
Continued fractions

ABSTRACT

Marcinkiewicz–Zygmund laws with convergence rates are established here for a class of strictly stationary ergodic sequences.

1. Introduction and result

Let $\{X_k\}_{k\in\mathbb{Z}}$, $\mathbb{Z}=\{\ldots,-1,0,1,2,\ldots\}$, be a strictly stationary sequence defined on a probability space $(\Omega,\mathcal{F},P)$ taking values on the real line $\mathbb{R}$ and $\mathcal{F}_k^m$ be the σ-field generated by $X_k, X_{k+1},\ldots,X_m$. Denote by T the usual shift operator on $\mathbb{R}^{\mathbb{Z}}$, i.e., for $\omega := (\omega_k; k\in\mathbb{Z})\in\mathbb{R}^{\mathbb{Z}}$, the element $T\omega\in\mathbb{R}^{\mathbb{Z}}$ is given by $(T\omega)_k=\omega_{k+1}, k\in\mathbb{Z}$. We say that $\{X_k\}$ is ergodic if T is ergodic. For $n\in\mathbb{N}=\{1,2,\ldots\}$ set

$$\varphi_n=\sup\bigl\{\bigl|P(B|A)-P(B)\bigr|;\ P(A)>0,\ A\in\mathcal{F}_{-\infty}^0,\ B\in\mathcal{F}_n^{\infty}\bigr\}.$$

$\{X_k\}$ is said to be uniformly strong mixing or φ-mixing if $\varphi_n\to_n 0$ (cf. [5]). Let $S_n=\sum_{k=1}^n X_k$ and for $0<r<2$ set $b=b(r)=0$ or $E[X_1]$ according as $r<1$ or $r\geqslant 1$. Recall, that a sequence $\{Y_n\}$ is bounded in probability (b.i.p.) if for each $\epsilon>0$ one can find $M>0$ with $\sup_n P[|Y_n|>M]<\epsilon$. It is well known that if $E[|X_1|^r]<\infty$, $r\in(1,2]$, then $\{n^{-\frac{1}{r}}(S_n-nE[X_1])\}$ is b.i.p. for sequences of random variables centered at conditional expectations given their preceding sums (cf. [1, Theorem 2]) and functionals of uniformly ergodic Markov chains or of digits of continued fraction expansion (cf. [22]).

The main result is a generalization of Théorème 9 in [16] and Theorem 1 in [2].

Theorem 1. *Suppose $r\in(0,2)$ and $\{X_k\}$ is a strictly stationary ergodic sequence such that $m=\inf\{n\in\mathbb{N}\mid\varphi_n<1\}<\infty$. Then the following conditions are equivalent*

$$n^{-\frac{1}{r}}S_n\to 0\quad \textit{almost surely}, \tag{1.1}$$

$$E\bigl[|X_1|^r\bigr]<\infty,\qquad b=0\quad\textit{and}\quad \bigl\{n^{-\frac{1}{r}}S_n\bigr\}\ \textit{is b.i.p.}, \tag{1.2}$$

$$\sum_{k=1}^{\infty}k^{-1}P\Bigl[\max_{1\leqslant i\leqslant k}|X_i|>\epsilon k^{\frac{1}{r}}\Bigr]<\infty\quad\textit{for any }\epsilon>0,\qquad b=0\quad\textit{and}\quad\bigl\{n^{-\frac{1}{r}}S_n\bigr\}\ \textit{is b.i.p.}, \tag{1.3}$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.10.011

$$\sum_{k=1}^{\infty} k^{-1} P\Big[\max_{1\leqslant i\leqslant k} |S_i| > \epsilon k^{\frac{1}{r}}\Big] < \infty \quad \textit{for any } \epsilon > 0, \tag{1.4}$$

$$\sum_{k=1}^{\infty} k^{-1} P\big[|S_k| > \epsilon k^{\frac{1}{r}}\big] < \infty \quad \textit{for any } \epsilon > 0. \tag{1.5}$$

Theorem 1 generalizes Corollary 8.3.3 in [13] (see also [22, Theorem 3.1]) with (1.3), (1.5) (here b.i.p. replaces $\sum_{k=1}^{\infty}\sqrt{\varphi_{2^k}} < \infty$). Moreover, Theorem 1 completes Theorem 1.1(a), (b), (c) in [18] for $p = r$, $\alpha = \frac{1}{r}$ with (1.5). In particular, in the case $r \in (0, 1]$, by the Markov and triangle inequalities $\{n^{-\frac{1}{r}}(S_n - nb)\}$ is b.i.p. Therefore for strictly stationary ergodic sequences with $\lim_n \varphi_n < 1$ the Kolmogorov (cf. [11, p. 67 (2nd English ed.); pp. 110–112 (2nd Russian ed.)]) and the Baum–Katz (cf. [2, Theorem 1]) for $t \in (0, 1]$ and for $t \in (1, 2)$, if $\sup_n \frac{E|S_n - nb|^t}{n} < \infty$, theorems are also true. Nevertheless, it appears to be unknown whether $\sup_n \frac{E|S_n - nb|^r}{n} = \infty$, $r > 1$, is possible under the conditions of Theorem 1. Note also that there exists a bounded and centered strictly stationary sequence such that (1.1) does not hold with $r > 1$ (cf. [3, Proposition 2.2]).

The proof of Theorem 1 relies on summation by parts (cf. [10, p. 322]) dependent Borel 0–1 criterion (cf. [13, p. 199]), symmetrization inequalities (cf. [14, p. 247]), variants of Cauchy's condensation principle (cf. [10, p. 121], [15]), dependent versions of the Hoffmann-Jørgensen (Proposition 1) and the Lévy (Proposition 2) inequalities and Berbee's decomposition of strictly stationary sequences (cf. [4]).

For an application of the main result denote by $a_n(x)$ the digits in the simple non-terminating continued fraction expansion of an irrational number $x \in (0, 1]$ (cf. [9, p. 4]). Let $\mathcal{B}$ be the σ-algebra of Borel subsets of $(0, 1]$ and P denote the Gauss measure: $\mathrm{P}(A) = (\ln 2)^{-1}\int_A (x+1)^{-1}\,dx$, $A \in \mathcal{B}$. It is well known that the random sequence $\{a_n\}$ defined on the probability space $((0, 1], \mathcal{B}, \mathrm{P})$ is strictly stationary exponentially fast φ-mixing sequence (cf. [9, pp. 17 and 49]).

Corollary 1. *Suppose that f is a Borel function and $r \in (0, 2)$. Then these statements are equivalent*:

(i) $n^{-\frac{1}{r}}\sum_{k=1}^{n} f(a_k) \to 0$ *almost surely*;
(ii) $\mathrm{E}[|f(a_1)|^r] < \infty$ *and* $b = 0$;
(iii) $\sum_{k=1}^{\infty} k^{-1}\mathrm{P}[\max_{1\leqslant i\leqslant k} |f(a_i)| > \epsilon k^{\frac{1}{r}}] < \infty$ *for any* $\epsilon > 0$ *and* $b = 0$;
(iv) $\sum_{k=1}^{\infty} k^{-1}\mathrm{P}[\max_{1\leqslant i\leqslant k} |\sum_{\nu=1}^{i} f(a_\nu)| > \epsilon k^{\frac{1}{r}}] < \infty$ *for any* $\epsilon > 0$;
(v) $\sum_{k=1}^{\infty} k^{-1}\mathrm{P}[|\sum_{i=1}^{k} f(a_i)| > \epsilon k^{\frac{1}{r}}] < \infty$ *for any* $\epsilon > 0$.

In the next section of this note some preliminary results required for the proofs in the last section are collected.

2. Preliminaries

Here is a collection of different results that will be used in the proof of Theorem 1 and Corollary 1. Let $X_1, X_2, \ldots, X_k, \ldots$ be a random sequence defined on a probability space $(\Omega, \mathcal{F}, P)$. Denote its independent copy by $\tilde{X}_1, \tilde{X}_2 \ldots$, and its symmetrized sequence $X_1 - \tilde{X}_1, X_2 - \tilde{X}_2, \ldots$, by $\hat{X}_1, \hat{X}_2, \ldots$ while in the case of stationarity by $X_1^*, X_2^*, \ldots$, its i.i.d. associated sequence (all sequences are sharing the same probability space). Set

$$Z_n = \max_{1\leqslant k\leqslant n} |S_k|, \qquad \hat{S}_n = \sum_{k=1}^{n} \hat{X}_k, \qquad \hat{Z}_n = \max_{1\leqslant k\leqslant n} |\hat{S}_k|,$$

$$M_n = \max_{1\leqslant k\leqslant n} |X_k|, \qquad \hat{M}_n = \max_{1\leqslant k\leqslant n} |\hat{X}_k|, \qquad M_n^* = \max_{1\leqslant k\leqslant n} |X_k^*|.$$

Since, in this section we also deal with non-stationary sequences we need to redefine φ_n as

$$\varphi_n = \sup_{J\in\mathbb{Z}} \big\{\big|P(B|A) - P(B)\big|;\ A \in \mathcal{F}_{-\infty}^{J},\ B \in \mathcal{F}_{J+n}^{\infty}\big\}.$$

By Theorem 5.2 in [5] the symmetrized coefficient $\hat{\varphi}_n$ of the sequence $\{\hat{X}_k\}$ satisfies

$$\hat{\varphi}_n \leqslant 1 - (1 - \varphi_n)^2, \tag{2.6}$$

for every $n \geqslant 1$.

The following proposition is a generalization of the Hoffmann-Jørgensen inequality (cf. [7], [12, p. 155]) and its proof is a modification of the proof of Lemma on p. 155 in [17] with the window of the size m deleted (cf. [23]).

Proposition 1. *Suppose $n, m \in \mathbb{N}$, $n > m$. For any positive s, t, u*

$$P[Z_n > s + 2t + u] \leqslant P[mM_n > u] + \big(\varphi_m + P[Z_n > t]\big) P[Z_n > s].$$

The following is a variant of the Lévy inequality and its proof requires the window of size $m-1$ removed (cf. [13, p. 192], [23]).

Proposition 2. *Suppose $n > m \geqslant 1$ and $\varphi_m < \frac{1}{2}$ and $\mathcal{L}(S_n - S_k)$ are symmetric for $n > k \geqslant 1$. Then for $t > 0$*

$$P\Big[|S_n| + (m-1) \max_{1 \leqslant i \leqslant n} |X_i| > t\Big] \geqslant \left(\frac{1}{2} - \varphi_m\right) P\Big[\max_{1 \leqslant k \leqslant n-m+1} |S_k| > t\Big]. \tag{2.7}$$

The statement below generalizes Proposition 6.8 in [12] and follows from Proposition 1 (cf. [23]).

Proposition 3. *Suppose $p > 0, n > m \geqslant 1, \tau \in (0,1)$ and*

$$t_\tau = \inf\Big\{t > 0;\, \varphi_m + P\Big[\max_{m \leqslant k \leqslant n} |S_k| > t\Big] \leqslant 4^{-p}\tau\Big\}.$$

If $E[|X_1|^p] < \infty$ then

$$E\Big[\max_{1 \leqslant k \leqslant n} |S_k|^p\Big] \leqslant \frac{4^p}{1-\tau}\Big(m^p E\Big[\max_{1 \leqslant i \leqslant n} |X_i|^p\Big] + t_\tau^p\Big). \tag{2.8}$$

The following inequality allows to compare maxima of a strictly stationary sequence with maxima of its associated sequence (cf. [19, p. 298]).

Proposition 4. *Suppose $\{X_k\}$ is a strictly stationary sequence and $\varphi_m < 1$. For every $x \geqslant 0$ and every $n \geqslant m \geqslant 1$*

$$(1 - \varphi_m) P\big[M^*_{\lfloor \frac{n}{m} \rfloor} > x\big] \leqslant P[M_n > x] \leqslant m(1 + \varphi_m) P\big[M^*_{\lfloor \frac{n}{m} \rfloor + 1} > x\big].$$

The next statement follows from Proposition 4 and the Cauchy condensation principle (cf. [21, p. 53]).

Proposition 5. *Suppose that $r > 0$ and $\{X_k\}$ is a strictly stationary sequence such that $\varphi_m < 1$ for some $m \geqslant 1$. Then for any $\epsilon > 0$ the following statements are equivalent*:

$$\sum_{k=1}^{\infty} \frac{1}{k} P\big[M_k^r > \epsilon k\big] < \infty; \qquad \sum_{k=1}^{\infty} P\big[M_{2^k}^r > \epsilon 2^k\big] < \infty; \qquad E\big[|X_1|^r\big] < \infty.$$

3. Proofs

Proof of Theorem 1. Fix $r \in (0,2)$ and set $c_n = n^{\frac{1}{r}}$. Assume, for the time being, that $\varphi_n \to 0$. Suppose (1.1) holds. For $n > 1$

$$\frac{|\hat{X}_n|^r}{n} \leqslant C_r \frac{|\hat{S}_n|^r}{n} + C_r \frac{|\hat{S}_{n-1}|^r}{n-1} \cdot \frac{n-1}{n},$$

where $\log_2 C_r = \max\{0, r-1\}$ (cf. [14, p. 155]), hence by the Borel 0–1 criterion for φ-mixing (cf. [13, p. 199])

$$E\big[|\hat{X}_1|^r\big] \leqslant \sum_{k=0}^{\infty} P\big[|\hat{X}_1|^r \geqslant k\big] < \infty \tag{3.9}$$

and by an argument on p. 243 in [14] it follows that $E[|X_1|^r] < \infty$. Moreover, Proposition 3 and (2.7) yield $n^{-1}E[|\hat{S}_n|^r] \to 0$. Therefore, for $r \in [1,2)$, by Lemma 4 in [1]

$$(2n)^{-1} E\big[|\hat{S}_n|^r\big] \leqslant n^{-1} E\big[\big|S_n - nE[X_1]\big|^r\big] \leqslant n^{-1} E\big[|\hat{S}_n|^r\big] \to 0 \tag{3.10}$$

so that by (1.1) $E[X_1] = 0$ and we arrive to (1.2).

If the condition (1.2) holds then Proposition 5 entails (1.3).

Assume (1.3) holds true and, without loosing generality, that $\epsilon = 1$. Set $X_{ki} = X_i I_{[|X_i| \leqslant c_k]}$, $Y_{ki} = X_i I_{[|X_i| > c_k]}$, $\hat{S}_{ki} = \sum_{\nu=1}^{i} \hat{X}_{k\nu}$, $\hat{T}_{ki} = \sum_{\nu=1}^{i} \hat{Y}_{k\nu}$, $\hat{Z}_{k\nu} = \max_{1 \leqslant i \leqslant \nu} |\hat{S}_{ki}|$. We have

$$\sum_{k=1}^{\infty}\frac{1}{k}P\big[|\hat{Z}_k|^r>k\big]\leqslant\sum_{k=1}^{\infty}\frac{1}{k}P\big[2C_r|\hat{Z}_{kk}|^r>k\big]+\sum_{k=1}^{\infty}\frac{1}{k}P\Big[2C_r\max_{1\leqslant i\leqslant k}|\hat{T}_{ki}|^r>k\Big]=I+II.$$

By (1.3), (2.7), Proposition 5 and

$$\sum_{k=1}^{\infty}\frac{1}{k}P\big[4C_r|\hat{T}_{kk}|^r>k\big]\leqslant 2\sum_{k=1}^{\infty}\frac{1}{k}P\left[8C_r\left|\sum_{\nu=1}^{k}Y_{k\nu}\right|^r>k\right]\leqslant 2\sum_{k=1}^{\infty}P\big[|X_1|^r\geqslant k\big]<\infty$$

we get that the series II converges. For the series I observe that by Proposition 3 with $p=2$ and $\tau=0.5$ and by Proposition 4

$$\sum_{k=1}^{\infty}\frac{1}{k}P\big[2C_r|\hat{Z}_{kk}|^r>k\big]\leqslant\sum_{k=1}^{\infty}\frac{1}{k^{1+\frac{2}{r}}}(2C_r)^{\frac{2}{r}}E\big[\hat{Z}_{kk}^2\big]\leqslant 2^5(2C_r)^{\frac{2}{r}}\sum_{k=1}^{\infty}\frac{1}{k^{1+\frac{2}{r}}}\Big(m^2E\Big[\max_{1\leqslant i\leqslant k}|\hat{X}_{ki}|^2\Big]+t_{0.5}^2\Big)$$

$$\leqslant 2^9(2C_r)^{\frac{2}{r}}\sum_{k=1}^{\infty}\frac{1}{k^{\frac{2}{r}}}m^3E\big[|\hat{X}_{k1}|^2\big]+2^5(2C_r)^{\frac{2}{r}}\sum_{k=1}^{\infty}\frac{1}{k^{1+\frac{2}{r}}}t_{0.5}^2=I_1+I_2,$$

where

$$t_{0.5}=\inf\left\{t>0;\ \hat{\varphi}_m+P\left[\max_{m\leqslant i\leqslant k}\left|\sum_{\nu=1}^{i}\hat{X}_{k\nu}\right|^r>tk\right]\leqslant 2^{-5}\right\}.$$

The series I_1 converges since $(\frac{2}{r}-1)\sum_{k\geqslant\nu}\frac{1}{k^{\frac{2}{r}}}\sim\frac{1}{\nu^{\frac{2}{r}-1}}$ (cf. [24, Theorem 8.7]) ($a_n\sim b_n$ means $\lim_n\frac{a_n}{b_n}=1$) and

$$\frac{1}{8}\sum_{k=1}^{\infty}\frac{1}{k^{\frac{2}{r}}}E\big[|\hat{X}_{k1}|^2\big]\leqslant\sum_{k=1}^{\infty}\frac{1}{k^{\frac{2}{r}}}E\big[|X_{k1}|^2\big]$$

$$\leqslant\sum_{k=1}^{\infty}\frac{1}{k^{\frac{2}{r}}}\sum_{\nu=1}^{k}\nu^{\frac{2}{r}}P\big[\nu-1\leqslant|X_1|^r<\nu\big]=\sum_{\nu=1}^{\infty}\nu^{\frac{2}{r}}P\big[\nu-1\leqslant|X_1|^r<\nu\big]\sum_{k=\nu}^{\infty}\frac{1}{k^{\frac{2}{r}}}$$

$$\leqslant C\sum_{\nu=1}^{\infty}\nu P\big[\nu-1\leqslant|X_1|^r<\nu\big]<\infty,\quad\text{for some } C<\infty.$$

For the series I_2 it is enough to prove that $t_{0.5}<\infty$, i.e., that

$$\left\{k^{-1}\max_{1\leqslant i\leqslant k}\left|\sum_{\nu=1}^{i}\hat{X}_{k\nu}\right|^r\right\}\text{ is bounded in probability.}$$

But this follows from (2.7) and observation that by the weak symmetrization inequalities (cf. [14, p. 245]) the sequences

$$\Big\{k^{-1}\max_{1\leqslant i\leqslant k}|\hat{X}_{ki}|^r\Big\},\qquad\left\{k^{-1}\left|\sum_{\nu=1}^{k}\hat{X}_{k\nu}\right|^r\right\},$$

are bounded in probability. This proves (1.4) for the sequence $\{\hat{X}_k\}$ and $\epsilon=1$. It is easy to see that for $0<\epsilon\neq 1$ the proof is the same. In particular $c_n^{-1}\hat{S}_n\to 0$ in probability and Proposition 3, (2.7), (3.10), (1.3) entails $n^{-1}E[|S_n|^r]\to 0$ which in turn enforces $c_k^{-1}\max_{1\leqslant i\leqslant k}|\mathrm{median}(S_i)|\to 0$. Whence by symmetrization inequalities (cf. [14, p. 247]) we get (1.4).

The condition (1.4) yields (1.5) trivially.

Suppose that (1.5) holds and set $u_k(\epsilon)=P[|S_k|^r>\epsilon k]$. Thus by stationarity for $0\leqslant\nu<2^k$, $k\geqslant 0$,

$$u_{2^{k+1}}(C_r\epsilon)=P\big[|S_{2^{k+1}}|^r>C_r\epsilon 2^{k+1}\big]=P\big[|S_{2^k-\nu}+S_{2^{k+1}}-S_{2^k-\nu}|^r>C_r\epsilon 2^{k+1}\big]$$

$$\leqslant P\big[C_r|S_{2^k-\nu}|^r+C_r|S_{2^{k+1}}-S_{2^k-\nu}|^r>C_r\epsilon\big(\big(2^k-\nu\big)+\big(2^k+\nu\big)\big)\big]$$

$$\leqslant P\big[|S_{2^k-\nu}|^r>\epsilon\big(2^k-\nu\big)\big]+P\big[|S_{2^k+\nu}|^r>\epsilon\big(2^k+\nu\big)\big]=u_{2^k-\nu}(\epsilon)+u_{2^k+\nu}(\epsilon).$$

Since $2^k = \lfloor\frac{2}{3}2^k\rfloor + \lfloor\frac{1}{3}2^k\rfloor + 1$ for $k \geqslant 0$, by the hypothesis

$$\infty > \sum_{k\geqslant 1}\frac{u_k(\epsilon)}{k} + \sum_{k\geqslant 0}\frac{u_{2^k}(\epsilon)}{2^k} = \sum_{k\geqslant 0}\sum_{i=2^k-\lfloor\frac{2^k}{3}\rfloor}^{2^k+\lfloor\frac{2^k}{3}\rfloor}\frac{u_i(\epsilon)}{i} + \sum_{k\geqslant 0}\frac{u_{2^k}(\epsilon)}{2^k} = \sum_{k\geqslant 0}\sum_{0\leqslant 3\nu<2^k}\left(\frac{u_{2^k-\nu}(\epsilon)}{2^k-\nu} + \frac{u_{2^k+\nu}(\epsilon)}{2^k+\nu}\right)$$

$$\geqslant \sum_{k\geqslant 0}\sum_{0\leqslant 3\nu<2^k}\frac{u_{2^k-\nu}(\epsilon)+u_{2^k+\nu}(\epsilon)}{2^k(1+\frac{1}{3})} \geqslant \sum_{k\geqslant 0}\sum_{0\leqslant 3\nu<2^k}\frac{u_{2^{k+1}}(C_r\epsilon)}{2^k(1+\frac{1}{3})} > \sum_{k\geqslant 0}\frac{u_{2^{k+1}}(C_r\epsilon)}{2^k(1+\frac{1}{3})}\frac{2^k}{3} = \frac{1}{4}\sum_{k\geqslant 1}u_{2^k}(C_r\epsilon)$$

(cf. [15, p. 356]). Therefore $\frac{|S_{2^n}|^r}{2^n} \to 0$ almost surely. Because ($S_0 = 0$)

$$\sum_{k=1}^{\infty}\frac{1}{k}P\big[|X_k| > \epsilon k^{\frac{1}{r}}\big] = \sum_{k=1}^{\infty}\frac{1}{k}P\big[|S_k - S_{k-1}| > \epsilon k^{\frac{1}{r}}\big]$$
$$\leqslant \sum_{k=1}^{\infty}\frac{1}{k}P\left[|S_k| > \frac{1}{2}\epsilon k^{\frac{1}{r}}\right] + \sum_{k=2}^{\infty}\frac{2}{k-1}P\left[|S_{k-1}| > \frac{1}{2}\epsilon(k-1)^{\frac{1}{r}}\right] < \infty$$

thus (1.5) and Proposition 5 imply $2^{-n}M_{2^n}^r \to 0$ almost surely. Moreover, by symmetrization inequalities $\frac{|\hat{S}_{2^n}|^r}{2^n} \to 0$ and $2^{-n}\hat{M}_{2^n}^r \to 0$ almost surely. Hence, by (2.7), stationarity and the Borel–Cantelli lemma

$$U_k = \max_{2^{k-1}<i\leqslant 2^k} 2^{-k}|\hat{S}_i - \hat{S}_{2^{k-1}}|^r \to 0 \quad \text{almost surely.}$$

For $2^{k-1} < n \leqslant 2^k$ we have

$$\frac{|\hat{S}_n|^r}{n} = n^{-1}\big|(\hat{S}_n - \hat{S}_{2^{k-1}}) + \hat{S}_{2^{k-1}}\big|^r \leqslant C_r\left(\frac{2^k}{n}U_k + \frac{2^{k-1}}{n}2^{-k+1}|\hat{S}_{2^{k-1}}|\right) \leqslant C_r\big(2U_k + 2^{-k+1}|\hat{S}_{2^{k-1}}|\big).$$

It follows that

$$\frac{|\hat{S}_n|^r}{n} \to 0 \quad \text{almost surely.} \tag{3.11}$$

Further, since $\frac{|S_{2^n}|^r}{2^n} \to 0$ almost surely, thus by (3.11), (2.7), (2.8) and (3.10) for $r \in [1,2)$, $E[X_1] = 0$ and $\frac{|S_n|^r}{n} \to 0$ in probability. Therefore $\mathrm{median}(\frac{|S_n|^r}{n}) \to 0$ and (3.11) with symmetrization inequalities yield (1.1).

Now, assume that T is ergodic, $\varphi_m < 1$ and $\lim_n \varphi_n \neq 0$ (see (1.11), Remark 2(b) on p. 124 and Theorem 4.1(3c) in [5]). By Theorem 4.2 in [5] (cf. [4, p. 292]) there exists $1 < p < \infty$ such that the invariant σ-field $\mathcal{I}$ of T^p is purely atomic with (modulo null sets) p atoms $E, TE, T^2E, \dots, T^{p-1}E$. Further, if we set $P_E(A) = P(A|E)$, $E \in \mathcal{I}$, $A \in \mathcal{F}_{-\infty}^{\infty}$,

$$\varphi_n^E = \sup\big\{\big|P_E(B|A) - P_E(B)\big|;\ P_E(A) > 0,\ A \in \mathcal{F}_{-\infty}^0,\ B \in \mathcal{F}_n^{\infty}\big\}$$

then $W_k = X_{(k-1)p+1} + X_{(k-1)p+2} + \dots + X_{kp}$ satisfies $\lim_n \varphi_n^E(\{W_k\}) = 0$ and $\varphi_m^E(\{W_k\}) < 1$. Therefore conditions (1.1)–(1.5) are equivalent for $\{W_k\}$ under P_E. On the other hand for $k > p$

$$S_k = W_1 + \dots + W_{\lfloor\frac{k}{p}\rfloor} + (S_k - W_{\lfloor\frac{k}{p}\rfloor}) = W_1 + \dots + W_{\lfloor\frac{k}{p}\rfloor} + R_k, \quad |R_k| \leqslant p\max_{k-p<i\leqslant k}|X_i|. \tag{3.12}$$

Thus, by (3.12), the condition (1.1) holds iff $n^{-\frac{1}{r}}|W_1 + \dots + W_n| \to 0$ P_E-a.s., say (1.1)(E) holds for every $E \in \mathcal{I}$. Now, assuming (1.2)(E) we get similarly that $\{n^{-\frac{1}{r}}S_n\}$ is b.i.p., $pEX_1 = EW_1 = 0$ and $E|W_1|^r < \infty$. Hence

$$E\big|\hat{W}_L - \hat{W}_L(T)\big|^r = E|\hat{X}_1 - \hat{X}_{L+1}|^r < \infty,$$

where L is such that $\hat{\varphi}_L^E < 0.5$. Furthermore,

$$P_E\big[|\hat{X}_1 - \hat{X}_{L+1}| > x^{\frac{1}{r}}\big] \geqslant P_E\big[\hat{X}_1 - \hat{X}_{L+1} > x^{\frac{1}{r}}\big] \geqslant P_E\big[\hat{X}_1 > x^{\frac{1}{r}};\, \hat{X}_{L+1} \geqslant 0\big]$$
$$\geqslant P_E\big[\hat{X}_1 > x^{\frac{1}{r}}\big]\big(0.5 - \hat{\varphi}_L^E\big)$$

and

$$P_E\big[|\hat{X}_1 - \hat{X}_{L+1}| > x^{\frac{1}{r}}\big] \geqslant P_E\big[\hat{X}_{L+1} - \hat{X}_1 > x^{\frac{1}{r}}\big] \geqslant P_E\big[-\hat{X}_1 > x^{\frac{1}{r}};\, \hat{X}_{L+1} \geqslant 0\big]$$
$$\geqslant P_E\big[\hat{X}_1 < -x^{\frac{1}{r}}\big]\big(0.5 - \hat{\varphi}_L^E\big)$$

so that

$$2P_E\big[|\hat{X}_1 - \hat{X}_{L+1}| > x^{\frac{1}{r}}\big] \geqslant P_E\big[|\hat{X}_1| > x^{\frac{1}{r}}\big]\big(0.5 - \hat{\varphi}_L^E\big).$$

Consequently, $E[|\hat{X}_1|^r] < \infty$ and $E[|X_1|^r] < \infty$. Thus we proved (1.1) $\Rightarrow$ (1.2). Step (1.2) $\Rightarrow$ (1.3) does not need φ-mixing so assume (1.3). It is easy to see that $\{n^{-\frac{1}{r}} W_n\}$ is b.i.p. and $\sum_{k=1}^{\infty} k^{-1} P_E[\max_{1\leqslant i\leqslant k} |W_i| > \epsilon k^{\frac{1}{r}}] < \infty$ so (1.4)(E) holds. Now, by (3.12) the condition (1.4) is satisfied as well. Further, since φ–mixing is not required and (1.4) $\Rightarrow$ (1.5) so assume (1.5). Clearly the latter entails (1.5)(E) so we have $n^{-\frac{1}{r}} W_n \to 0$, P_E almost surely. By (3.12) it is evident that $n^{-\frac{1}{r}} S_n \to_{a.s.} 0$, too. The proof is completed. □

Proof of Corollary 1. By Corollary 1.3.15 in [9] ψ_n coefficient for the sequence $\{f(a_k)\}$ defined by

$$\psi_n = \sup_{k\in\mathbb{N}} \sup\left\{\left|\frac{P(A\cap B)}{P(A)P(B)} - 1\right|;\ P(A)P(B) > 0,\ A \in \mathcal{F}_1^k,\ B \in \mathcal{F}_{n+k}^{\infty}\right\}$$

fulfills

$$\psi_1 \leqslant 2\ln 2 - 1 < 0.387, \qquad \psi_2 \leqslant \frac{\pi^2}{6}\ln 2 - 1 < 0.141,$$

and

$$\psi_n \leqslant \frac{1}{2}(\ln 2)(3.5 - 2\sqrt{2})^{n-2}, \quad n \geqslant 3.$$

Because $\psi_n \geqslant 2\varphi_n$, $n \geqslant 1$ (cf. [5, (1.11)]), therefore $\varphi_1 < 1$ and the dependence condition in Theorem 1 is satisfied. Thus we need to prove that $\{n^{-\frac{1}{r}} S_n\}$ is b.i.p. If $\mathrm{E}[f^2(a_1)] < \infty$ then the CLT holds (cf. [8, Theorem 18.5.2], and [6, Remark 5]) and since $n^{\frac{1}{r}} > n^{\frac{1}{2}}$ so $\{n^{-\frac{1}{r}} S_n\}$ is b.i.p. Now assume $\mathrm{E}[f^2(a_1)] = \infty$. Note that by Proposition A3.1 on p. 326 in [9], for some $C > 0$ and $\rho \in (0,1)$

$$\big|\mathrm{Cov}\big[f(a_1)I_{[|f(a_1)|\leqslant n^{\frac{1}{r}}]} f(a_k)I_{[|f(a_k)|\leqslant n^{\frac{1}{r}}]}\big]\big| \leqslant C\rho^k \mathrm{E}^2\big[\big|f(a_1)\big|I_{[|f(a_1)|\leqslant n^{\frac{1}{r}}]}\big].$$

Because $\lim_{x\to\infty} \mathrm{E}[|f(a_1)|^2 I_{[|f(a_1)|\leqslant x]}] = \infty$ thus

$$\mathrm{E}^2\big[\big|f(a_1)\big|I_{[|f(a_1)|\leqslant n^{\frac{1}{r}}]}\big] = o\big(\mathrm{E}\big[\big|f(a_1)\big|^2 I_{[|f(a_1)|\leqslant n^{\frac{1}{r}}]}\big]\big)$$

(cf. (2.6.14) in [8]) and therefore

$$\begin{aligned}\mathrm{Var}\left[\sum_{k=1}^n f(a_k)I_{[|f(a_k)|\leqslant n^{\frac{1}{r}}]}\right] &= n\mathrm{Var}\big[f(a_1)I_{[|f(a_1)|\leqslant n^{\frac{1}{r}}]}\big] + 2\sum_{k=2}^n (n-k+1)\mathrm{Cov}\big[f(a_1)I_{[|f(a_1)|\leqslant n^{\frac{1}{r}}]} f(a_k)I_{[|f(a_k)|\leqslant n^{\frac{1}{r}}]}\big] \\ &\sim n\mathrm{E}\big[\big|f(a_1)\big|^2 I_{[|f(a_1)|\leqslant n^{\frac{1}{r}}]}\big].\end{aligned}$$

Hence by the Lyapunov inequality (cf. [14, 9.3c, p. 156], [20, (1.9), p. 7])

$$\begin{aligned}&\mathrm{E}^2\left[n^{-1}\left|\sum_{k=1}^n \big(f(a_k)I_{[|f(a_k)|\leqslant n^{\frac{1}{r}}]} - \mathrm{E}\big[f(a_1)I_{[|f(a_1)|\leqslant n^{\frac{1}{r}}]}\big]\big)\right|^r\right] \\ &\quad\leqslant \mathrm{E}^r\left[n^{-\frac{2}{r}}\left|\sum_{k=1}^n \big(f(a_k)I_{[|f(a_k)|\leqslant n^{\frac{1}{r}}]} - \mathrm{E}\big[f(a_1)I_{[|f(a_1)|\leqslant n^{\frac{1}{r}}]}\big]\big)\right|^2\right] \\ &\quad\leqslant C_\rho^r\big(n^{1-\frac{2}{r}}\mathrm{E}\big[\big|f(a_1)\big|^2 I_{[|f(a_1)|\leqslant n^{\frac{1}{r}}]}\big]\big)^r \leqslant C_\rho^r\big(n^{\frac{r-2}{r}} n^{\frac{2-r}{r}}\mathrm{E}\big[\big|f(a_1)\big|^r I_{[|f(a_1)|\leqslant n^{\frac{1}{r}}]}\big]\big)^r = C_\rho^r\big(\mathrm{E}\big[\big|f(a_1)\big|^r I_{[|f(a_1)|\leqslant n^{\frac{1}{r}}]}\big]\big)^r.\end{aligned}$$

Since $n\mathrm{P}[|f(a_1)| > n^{\frac{1}{r}}] \to 0$ and

$$n^{1-\frac{1}{r}}\big|\mathrm{E}\big[f(a_1)I_{[|f(a_1)|\leqslant n^{\frac{1}{r}}]}\big]\big| = n^{1-\frac{1}{r}}\big|\mathrm{E}\big[f(a_1)I_{[|f(a_1)|> n^{\frac{1}{r}}]}\big]\big| \leqslant \mathrm{E}\big[\big|f(a_1)\big|^r I_{[|f(a_1)|> n^{\frac{1}{r}}]}\big]$$

thus the b.i.p. condition holds. □

Remark 1. By the proof of Theorem 1 in [23] the boundedness in probability of $\{n^{-\frac{1}{r}} S_n\}$ in conditions (1.2) and (1.3) can be replaced by $n^{-\frac{1}{r}} S_n \to 0$ in probability (cf. [12, Theorem 7.9 on p. 186]). Thus for a strictly stationary φ-mixing sequence $\{X_k\}$ the stability of $\{n^{-\frac{1}{r}} S_n\}$, $r \in (0,2)$, is strong if and only if $E[|X_1|^r] < \infty$. Moreover, if the latter holds then one can take $d_n = n^{1-\frac{1}{r}} b$ (cf. [11, pp. 61–67 (2nd English ed.) and pp. 88–112 (2nd Russian ed.)]).

Acknowledgment

I would like to thank anonymous referees for comments and suggestions that improved the presentation of the paper.

References

[1] B. von Bahr, C.-G. Esseen, Inequalities for the r-th absolute moment of a sum of random variables, $1 \leqslant r \leqslant 2$, Ann. Math. Stat. 36 (1965) 299–303.
[2] L.E. Baum, M. Katz, Convergence rates in the law of large numbers, Trans. Amer. Math. Soc. 120 (1965) 108–213.
[3] J. Baxter, R. Jones, M. Lin, J. Olsen, SLLN for weighted independent identically distributed random variables, J. Theoret. Probab. 17 (1) (2004) 165–181.
[4] H. Berbee, Periodicity and absolute regularity, Israel J. Math. 55 (1986) 289–304.
[5] R.C. Bradley, Basic properties of strong mixing conditions. A survey and some open questions, Probab. Surv. 2 (2005) 107–144.
[6] M.I. Gordin, On the behaviour of the variances of sums of random variables forming a stationary sequence, Theory Probab. Appl. 26 (3) (1971) 474–484.
[7] J. Hoffmann-Jørgensen, Sums of independent Banach space valued random variables, Studia Math. 52 (1974) 159–186.
[8] I.A. Ibragimov, Yu.V. Linnik, Independent and Stationary Sequences of Random Variables, Wolters-Noordhoff, Gröningen, 1971.
[9] M. Iosifescu, S. Kraaikamp, Metrical Theory of Continued Fractions, Mathematics and its Applications, vol. 547, Kluwer Academic Publishers, Dordrecht, 2002.
[10] K. Knopp, Theorie und Anwendung der unendlichen Reihen, Springer-Verlag, Berlin, 1964.
[11] A.N. Kolmogorov, Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer-Verlag, Berlin, New York, 1933; reprinted: Foundations of the Theory of Probability, 2nd ed., Chelsea Publ. Comp., New York, 1956; Fundamental Concepts of Probability Theory, 2nd Russian ed., Nauka, Moscow, 1974.
[12] M. Ledoux, M. Talagrand, Probability in Banach Spaces, Springer-Verlag, Berlin, 2002.
[13] Z.Y. Lin, C.R. Lu, Limit Theory for Mixing Dependent Random Variables, Mathematics and its Applications, vol. 378, Science Press, Beijing, 1996.
[14] M. Loève, Probability Theory, 2nd ed., Van Nostrand, New York, 1960.
[15] R. Lyons, Strong laws of large numbers for weakly correlated random variables, Michigan Math. J. 35 (1988) 353–359.
[16] J. Marcinkiewicz, A. Zygmund, Sur les fonctions indépendantes, Fund. Math. 29 (1937) 60–90.
[17] S.V. Nagaev, Probabilistic and moment inequalities for dependent random variables, Theory Probab. Appl. 45 (2000) 152–160.
[18] M. Peligrad, The r-quick version of the strong law for φ-mixing sequences, in: G.A. Edgar, L. Sucheston (Eds.), Almost Everywhere Convergence, Academic Press, New York, 1989, pp. 335–348.
[19] M. Peligrad, On Ibragimov–Iosifescu conjecture for φ-mixing sequences, Stochastic Process. Appl. 35 (1990) 293–308.
[20] V.V. Petrov, Limit Theorems of Probability Theory. Sequences of Independent Random Variables, Oxford Studies in Probability, vol. 4, The Clarendon Press, Oxford University Press, New York, 1995.
[21] W. Rudin, Principles of Mathematical Analysis, 2nd ed., McGraw-Hill, New York, 1964.
[22] Q.-M. Shao, Maximal inequalities for partial sums of ρ-mixing sequences, Ann. Probab. 23 (2) (1995) 948–965.
[23] Z.S. Szewczak, Marcinkiewicz laws with infinite moments, Acta Math. Hungar. 127 (2010) 64–84.
[24] A. Zygmund, Trigonometric Series, vol. I, Cambridge University Press, New York, 1959.